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Thinking on the Practical Plan of "the Monotonicity of Function"

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This study emphasized on practice of "the monotony of function". The author had made a practical plan and comment on seven phases of conceptual understanding of "the monotony of function". Through the process of teaching model, the student had better understanding of dynamic mathematics in symbolic mathematical format and representation. Results showed there was an urgent need to blend educational principle into practice.

***Key words*:** the monotony of, function, teaching situation, conceptual formation, action research.

Background

"The Monotonicity of Function" is not only an important mathematical concept in senior high school, but also an important character of function in mathematics. It plays an essential role in the middle school mathematics. At present, when teaching "the monotony of function", the teacher usually explains the definition verbally in middle school. Although the teacher may enlighten the students, there is a lack of active management of the students. There is especially a lack of constructive thinking of them. Essentially, the students lack "the constructive process" in their minds. By using this teaching method, some students may carry out meaningful learning; however, most students can only achieve to the mechanical learning level such as memorizing meaning and imitating exercise. In fact, many teachers desire to know how to apply the exploration method to the learning process of "the Monotone of Function" because this method can make students experience the process of constructive thinking. Teachers tried to solve this problem and some of them did carry out experiments, but the result is not satisfactory. Teachers feel it is difficult to cope with the exploration method.

The Basic Understandings of Teaching "the Monotony of Function"

Before learning this function content knowledge, the students have already learned many simple functions, such as linear function, quadratic function, inverse proportion function and so force. The students have known the variable definition and the mapping definition of the function, as well as the expression of the function. They have understood that the function can be depicted as a quantitatively. In mathematics learning, the significance of constructing one mathematics conception is to reveal its' essence such as common and invariable properties. The aim is to study the transformation laws of movement which various function models describe, i.e. common and invariable properties of these changing movement connections, which is the property of "invariability in changing". According to scientific studying on this thinking manner, it becomes an essential and logical activity to discuss some functions properties. Many function's properties with "the Monotone of Function" are easy to explore. Thus, the idea of bringing forward the project is formed.

Generally speaking, the existing two important processes in the construction of the function's monotone: construct its meaning and depict the meaning in symbolic math language by constructive thinking. Since students to find the meaning of the monotone of function by observing some function graphs is easy, it is relatively easy to construct and learn based on the former process. However, the difficult is how we can utilize the students' thought activities by using math language to depict the conception of the function's monotony: (a) How to use symbols to indicate "x increases" and "$f(x)$ increases"? (b) How to use symbols to indicate "$f(x)$ increases with the increase of x"?

The most difficult of using math symbols to indicate the above two mathema-

tics meanings lies in using mathematics symbols to depict dynamic mathematics objects. In junior middle school, students encounter only a little dynamic mathematics objects that are depicted by mathematics symbols. For example, when students learning the primary conception of function, they use $y=f(x)$ to indicate function y changes with the changes of independent variable x. As a result, at that time, most mathematics symbols that students encounter are to depict the static mathematics objects. Thus, the great disparity is the abstract thinking capabilities levels after the transformation from using mathematics symbols to depict static math objects and to depict dynamic math objects. Undoubtedly, it is a great challenge for students in senior high schools, and it is also the reason why it is complex to make a breakthrough to teach this difficult concept.

Teaching Design of "the Monotone of Function"

The "conceptual formation" of conceptual learning patterns is adopted to design the teaching of "the monotone of function" on the one hand is to reflect the guiding effects of contemporary educational theories and on the other hand is to center on solving "the monotony of function" the teaching and learning center of realistic math problems. It is an active attempt to use the dual principle ——"corres-

pondence of teaching and learning" and "correspondence of teaching and math".

We'll describe our design briefly and give some concise commentaries in the following.

Basic Ideas: With the guidance of the contemporary educational theories, we adopted the "conceptual formation" of conceptual teaching patterns to plan the teaching of "the monotone of function". The plan aims at constructing the knowledge sense by thinking, recognizing the methods of scientific research and developing students' thought, and attempting to show the formation process of knowledge, emphasizing teaching situations set creatively by teachers, and giving prominence to the process of students' initiative learning and recreating of the activity of math.

Pose Problem: The aim is to make students understand why they need to learn "the monotone of function". (It has been explained in the foregoing context.)

At the beginning of each mathematical class, the teacher should first ask the students:

"What should we study today?" That is "Why it is worth studying" and "How to think of the issue". However in reality, teachers generally focus on "what the specific details are". The result is that students only absorb knowled-

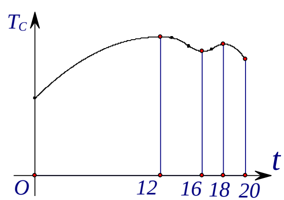
ge but do not know why they should do so, which in fact is that they do not know how to raise questions. The emphasis of proposing the issue is a real recognition on the implication and infiltration of scientific thoughts and methods.

Create Situation: adopt multi-media technology to design the state of functional dynamic changes and make students have complete sense perception on all the changes of images as well as other associated aspects. There are several functional graphs in the context: linear function, simple quadratic function, temperature changing chart of a certain day somewhere and the dynamic changes of the above-mentioned graphs with specific needs. These dynamic changes include graph ascentive or decentive, two spots moving on x axis and their corresponding function graph's changing and also x and corresponding' f(x) numerical tables in some monotone interval. All kinds of changes displayed in functional graphs reflect the essential characteristics of monotone function as far as possible and potentialize all sorts of situations removing the disturbance of non-essential attributes. Thus by perceiving the vivid situation, students can access to a wealth of representation and information and generate various associations.

As far as the mathematics teaching is concerned, learning and mastering the essence of mathematics content is most important. Therefore, the key to the teaching situation creation is the ability to reveal the nature of mathematics. There are different ways to create teaching situations. For the mathematics objects related with moving and changing, multimedia technology can be used to create correlative situations, which advantage is to display the moving process of mathematics objects vividly and intuitively and to reveal the characteristics of the object's various varying. This can help the students see visuals about the objects studied in situation and grasp the essence of the mathematical properties of the object. The associations with actual problem should be paid attentions during the situations creation processes.

Stimulation Stage: show students the dynamic changing process of functional raph (in Figure 1 to 4), make student fully observe the various changes of functional graph and organize a discussion. There should be more presenting of graphs and less language interpretation, especially those characteristics require students to find out should be left for students and let students observe and reflect by themselves. This not only embodies the main features of mathematics constructivism leaning, but also trains general hinking methods such as observation, association, comparison, analysis, synthesis, abstract, summarize and makes students to experience and apperceive the spirit of the mathematical thinking methods.

Analysis stage: based on observing the dynamic changes of a few function graphs (next page) mentioned above，students compare the function graphs from multi-perspectives, analyze the characteristics of each respective graph further and attempt to find the similarities and differences of them in the process.



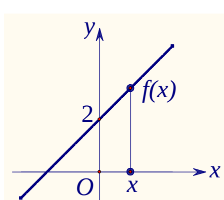
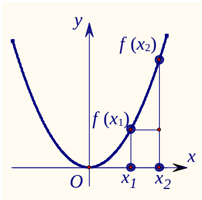


Figure 1: **Temperature changes in a day at a certain place.**

Figure 2: **y=x+2.**



x	0.96<1.08<1.26<1.75
y	1.16<1.47<1.97<2.53

Figure 3: **y= x^{2}.**

Figure 4: **y=x^2, x∈ (0, +∞).**

Cognitive psychological studies have shown that a person form the perception and concept through the connection between external clues (stimulus or some characteristics) and the internal intermediary process (meaning, ideas or concepts). In creating teaching situations of the monotone of function, different "function graphs" and "changing trend of function graphs" in perception are external stimulus, and the dynamic changes of graphs give prominence to the characteristics that students need to understand, so that these external stimulus and the meaning and concept of "ascentive" and "decentive" during the corresponding section will be linked up through the process of internal nerve or brain of perception, approach the goal designed by situation step by step, and eventually reach the desired goal .

Differentiation: To enable students to observe and discuss fully, put forward their views, contend with each other, and to differentiate some relatively common characteris-

tics or the nature of these graphs.

Conclusions: Figure 1: The function domain is the interval [0, 20] ;the graph is ascentive in [0, 12] ;the graph is decentive in [12, 16] ;the graph is ascentive in [16, 18] ;the graph is decentive in [18, 20] .

Figure 2: The function y=x+2, x∈**R**, the graph of the function is always ascentive, i.e. the graph is ascentive in the entire domain (-∞, +∞).

Figure 3: The function y=x^2, x∈**R** , y≥0 , the left branch of the graph is decentive in (-∞, 0) ; the right branch of graph is ascentive in (0, +∞) .

Figure 4: In (0, +∞), the corresponding value of the function y=x^2 increases with the increasing of the value of x.

For many students, there are varying degrees of difficulty in using the mathematical language to express the meaning of "left and right branch" . There is little or no similar study in junior high school , so they cannot adapt to using the mathematical symbols to express the visual graphics . When the students are picking up the characteristics, the expression is often not in place, some with improper language and some with misunderstandings. At this time the discussion and supplement between students are very important ,so need to be patient through student themselves cooperating and exchanging a relatively accurate statement.

Discuss the Problems : Aiming at different views of “ascentive”, “decentive”, “Increase”, “Decrease”, guide students to discuss a problem in real life: is the road between the school and the Gulou district up or down? (some students think that it is going up, and some students say that it is going down.)

Let the students think about the problems by themselves according to their different opinions why there exist different views? And then let them realize that whether it is ascentive or desentive depends on the direction; therefore, at first, we need to have the same identification of the direction.

It is necessary to make students discuss a series of problems. It is best that the questions are put forth by students themselves, or by students who are guided by teachers. Also, teachers can put forth the problems directly with a fundamental premise that teachers never work instead of students. The situation that the problems are put forth by the teachers appears when the students have problems but they can’t state them clearly.

Induct and Discuss: Whether the road is ascentive or decentive depends on the direction. There should be a same frame of reference, including references and directions. The research for moving object must be within the same frame of reference; otherwise, the same meaning of one object will have different expressions, which will lead to ambiguity.

For the increasing and decreasing parts of the graph of a function, what should be considered as the reference? Refer to the direction of x-axis. Why? This is because along the direction of x-axis, the value of x increases singly, whereas the graph of the function has both increasing and decreasing parts along this direction. In this way, it is consistent with the common sense of "increasing" and "decreasing" and thus no misunderstanding will be generated.

Necessary Instructions: The increasing or decreasing of the graphs shows that there is “invariability in changing” of functions, which is called “monotonic” in mathematics. They are called “monotone increasing” and “monotone decrease” respectively.

Classify: During the classifying process of conceptual formative learning, the mission is to guide the students to generalize the common features of several function images.

Summarize Features: There is a common characteristic of function image: it is either up or down in some parts of the domain.

At this teaching stage, students can already understand the meaning of the monotone function on the intuitional sense. However, in the processes of the leaning in mathematics, it is just the beginning of the researches that clear up the research subjects, namely the meaning of mathematics objects. And also the meanings of objects are expressed by symbols of mathematics language.

This is a kind of learning in mathematical language and a process of training, cultivating and developing of the mathematics thinking. At the perspective of educa-tion, the study of mathematics thinking is more important and more significant than knowledge learning.

Discuss the problems： both "ascentive" and "desentive" are ordinary language. It is imprecise to describe the mathematic characteristic of "monotone increasing" and "monotone decreasing" with the ordinary language . Then , can we use the math language to describe this characteristic of the function？If we could , how can we describe it？

This is a process which describes "monotone increasing" and "monotone decreasing" from using dynamic image to using static mathematical symbols . The student should have enough time to discuss adequately, find the method of abstract formulation in mathematics , and propound the hypothesis of the definition about "the monotone of function". The difficulty is how to express "ascentive" and "decentive" in math language .

After discussion, the students put forward a kind of form:

$$\begin{cases} \textit{ascensive: the function } f(x) \textit{ increasing along with the } x\textit{'s increasing} \\ \textit{degressive: the function } f(x) \textit{ decreasing along with the } x\textit{'s increasing} \end{cases}.$$

Through the further discussion , towards "the function *f*(*x*) increasing along with the *x*'s increasing", the student could easily put forward a formulation with graphical symbol:

$$\begin{cases} \textit{ascensive: } f(x) \textit{ increasing along with the } x\textit{'s increasing} \\ \textit{degressive: } f(x) \textit{ decreasing along with the } x\textit{'s increasing} \end{cases}$$

$$\xrightarrow[\text{"↗" "↘" to indicate}]{\text{use graphical symbol}} \begin{cases} \textit{ascensive: } x\nearrow \textit{ and } y=f(x)\nearrow \\ \textit{degressive: } x\nearrow \textit{ and } y=f(x)\searrow \end{cases}.$$

This process follows in gradual sequence. The two expressions transform ordinary language symbols to graphical symbols. They are surely more mathematical than the original word language. However, these graphical symbols of "↗ " and "↘ " also based on qualitative natural description, so this kind of description is not given by rule and line. Only by using the mathematic symbols which represent amount (such as algebraic symbol) to form a quantitative description, ambiguity can be eliminated and accuracy can be met.

Go on with this discussion: towards the graphical symbols of "↗ " and "↘ ", maybe different people have different association. In other words, it will still cause ambiguity. Experience proves that the amount is unlikely to cause ambiguity. Then can we use the symbol which represents quantities to describe "$x\nearrow$, $y=f(x)\nearrow$ "?

The discussion shows that "increase" or "↗ " must be the result of comparison of two variables. To indicate "increase", two variables with different numerical values are necessary. They can be denoted by $x_1 < x_2$, and consequently there are two corresponding function values f (x_1) and $f(x_2)$. There is also a magnitude relationship between them. It should be noticed that x_1 and x_2 should be picked both from the increasing part of the graph or the decreasing part of it. Through trying, the students could obtain the following expression in mathematic symbols：

$$\begin{cases} \text{ascensive: } x\nearrow \text{ and } y=f(x)\nearrow \\ \text{degressive: } x\nearrow \text{ and } y=f(x)\searrow \end{cases} \xrightarrow[\text{use mathematic symbol to denote "↘"}]{\text{use mathematic symbol to denote "↗"}}$$

$$\begin{cases}\text{ascensive: } x_1 < x_2 \text{ and } f(x_1) < f(x_2) \\ \text{degressive: } x_1 < x_2 \text{ and } f(x_1) > f(x_2)\end{cases}.$$

It is the same meaning that the mathematical expression "if $x_1 < x_2$, then $f(x_1) < f(x_2)$" is more exactly than the meaning of "x↗ , $y=f(x)$↗ ". It is still not a strict expression of the monotone of function, since the mathematical expression cannot reflect the random city of x_1 and x_2, and it cannot completely reflect the "invariability in changing" of *monotone increasing* and *monotone decreasing*. There is a little flaw and it may not be a bad thing, because this flaw can provide the material for the following verification. "Verification" is an indispensable way of thinking. Learning and understanding *verification* is very necessary for the students' thinking development and the scientific concept cognition.

Abstract: After the above completion, the students have the initial condition to create the hypothesis about the definition of the monotonicity of function. The definition reflects the substance of functions' monotonicity. Thus, the stage of abstraction is the stage of asking the conjecture about the substance of the functions' monotony.

Propose the hypothesis (definition of monotone function): Take $x_1, x_2 \in (a, b)$ and $x_1 < x_2$,If $f(x_1) < f(x_2)$ then the function f(x) is called a monotone increasing function in (a, b). Similarly Take $x_1, x_2 \in (a, b)$ and $x_1 < x_2$, If $f(x_1) > f(x_2)$ then the function f(x) is called a monotone decreasing function in (a, b).

Prove-test: The assumption is correct or not needs verification and the variant is the primary way of verification.

Re-observe: At this stage, the teacher ought to ask the students to observe the teaching situation again and again, compare the assumption of the definition with the teaching situation, and analyze the assumption. Then, they can find the flaw in the assumption of the definition and give some variant counterexamples. The teacher also can put forward some counterexample, if it is necessary.

To make variant form: Figure5 is the graph of $f(x)=x^2$. In the interval (a, b), take two numbers $x_1, x_2 \in (a, b)$. As shown in Figure5, $x_1 < x_2$ and $f(x_1) < f(x_2)$.

Is the function $f(x)=x^2$ a monotone increasing function the interval (a, b)?

Discuss the variant forms: As showed in Figure5, $f(x)=x^2$ isn't a monotone increasing function in the interval (a, b). What is wrong? In contrast with Figure2 and Figure5, the students can easily found the reason for this mistake is x_1 and x_2 only are special numbers in (a, b).

To reflect a characteristic that always exists, the two points x_1 and x_2, should be arbitrarily chosen from the interval (a, b), and for each$x_1<x_2$, here exists $f(x_1) < f(x_2)$. Then having the arbitrarily chosen x_1 and x_2, as well as the conditions $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, the monotonously increasing or decreasing characteristic could be accurately depicted.

In the practical teaching, it may difficult to verify the definition hypothesis. First, make negative examples to guide students is difficult. Second, instruct students to understand is difficult, because we can not reveal the nature of the monotonicity of function if there is no arbitrariness. The students may not be able to make the variational modalities. Teachers should avoid only following their previous design but regardless of students' thinking. Teachers should advocate students to find out and any thoughts and guided them properly. This way, not only protects the students' enthusiasm, but also opens up their thinking ability. So constructive teaching is a challenge not only to students but also to teachers.

Generalize stage: *This stage is based on the work of amending the hypothesis according to prove-test, and summarized to the general situation.*

Generalize conclusion: Let x_1 , $x_2 \in$ (a, b) and $x_1 < x_2$, if $f(x_1) < f(x_2)$, then f(x) is called Monotone Increasing Function . Let x_1 , $x_2 \in$ (a, b) and $x_1 < x_2$, if $f(x_1) > f(x_2)$, then f (x) is called Monotone Decreasing Function .

In most cases, the former definition is summarized by teacher and students together, and students generalize the latter by themselves. Let students to draw a summary by students is better than teacher, although this summary may imitativeness to some extent. On the one hand, imitation contains cognition and is a process of internalization and the interaction of internal cognition structures. On the other hand, students learn this purely mathematical formal language for the first time and use it to describe an object, thus a little imitative study is necessary.

Formalization: Here formalization process has been completed in fact.

Analysis of the Use of Some Educational Theories

The above-mentioned teaching plan is designed according to the learning mode of "concept formation" which is one mode of a large number of programs. "The seven phases of teaching model" may be not understood and used mechanically. The meaning of every stage is not absolute and their order may change. Some of the stages should be selected as needed instead using all of them. In fact, some stages of this design also contain the activities of other stages, so they should not be used rigidly.

It is necessary to establish a concept that or an education theory could be not treated dogmatically. The theory itself originated from the specific situation in which it is rooted, so the use of the theory can not divorce from the practice and should grasp the substance of the theory and actual needs.

The practice is composed by the mathematics instruction contents, the status of students' knowledge, the development of mathematical thinking, the background of students' real life, and students' cultural tradition. Good teaching results cannot be attained unless theory is combined organically with the practice.

Constructive learning depends upon two components: the proper situation set up by the teacher and the essential and inartificial contact built by students between the situation and the pertinent knowledge deposited in their cognitive structure. The former expresses "the teachers' creative instruction", namely to set up a situation creatively, and the latter expresses the students' creative learning, namely to re-create the meanings of mathematics objects.

The monotonicity of functions has broad contacts with knowledge, experience, thoughts and methods, for examples, with the movement of function graphs(increasing, decreasing), with the frame of reference or direction moving along the x-axis, with the range of variable (monotonicity is defined on a interval), with the concepts of functions (rules, domains, ranges), with mathematics language (For $x\nearrow$, $y=f(x)\nearrow$ or $y=f(x)\searrow$ .When $x_1<x_2$, $f(x_1)<f(x_2)$; when $x_1<x_2$, $f(x_1)>f(x_2)$), with logical experience (for any arbitrarily chosen $x\in(a, b)$, and, let $x_1<x_2$), with strategies to compare big with small (to subtract, to divide), with thought way and experience of mathematical proof (The previous prominent way is geometric proof based on theorems. Here the way is mostly to utilize definition, and to judge the factor's positive or negative property), and the like.

The above mentioned knowledge contributes to the learning of monotonicity city of functions and it is impossible to construct the meaning of monotonicity without such knowledge, experience, thoughts and methods. Meanwhile the construction of the understanding of monotonicity is also the application of themselves, which enables their meaning to be extended, enlarged and renewed. This is where the significance of the so-called bi-direction construction lies in.

Further Assimilation to the Meaning of Functions' Monotonicity

After the thinking operation is completed, which includes constructing meaning of concept and formalized definition, reflecting and analyzing concep-

tual meaning run proceed as essential phases to differentiate and integrate the meaning of concept, and these phases represent the process of further assimila-

tion. It is proved in practice an effective way to realize further assimilation with the help of variant form.

For example, variant form is useful to clarify anddeepen the cognition of function monotonicity. Suchquestion may be asked: Is the following assertion right?

Eg1 *Let quadratic function* $f(x)=x^2$*. Considering* $-1,2\in(-\infty,+\infty)$*, for* $-1<2$*,* $f(-1)<f(2)$*. Thus by the above,* $f(x)=x^2$ *is an increasing function on interval* $(-\infty,+\infty)$[4].

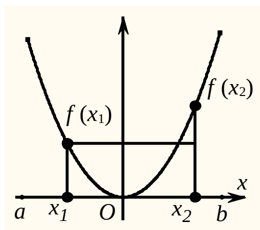


Figure 5: **Eg1**

Eg.2 *The function* $y=f(x)$, $x\in[0, +\infty]$*, if for any* $x_2>0$ $f(x_2)<f(0)$*, then the function is the monotonic decreeing function on* $(0,+\infty)$.

The variant form from the definition could be used to clarify and deepen the unders tanding of the monotone function interval. Try to ask the following questions of the variant form:

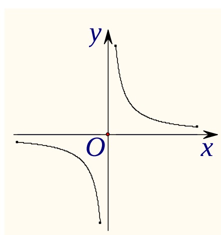


Figure 6: **Eg3**

Eg.3 *Is* $y=\dfrac{1}{x}$ *a decreasing function? If it is, then what is he monotone interval?*

Sometimes, the students will say the monotone interval is $(-\infty,+\infty)$, because they regard the function as a decreasing function in its whole definition domain according to the figure 6.

This kind of variant form is very edifying for the students to pprehend deeply the interval of the monotone function.

We can not construct the purport of a mathematical concept once for all. Actually, we must ruminate, differentiate, analyze and generalize the meanings of the concept continually in order to digest the essence of the mathematical concept after stepwise fermentation and integration.

Of course, only one lesson cannot reach this goal. It's impossible to enable the students to have a thorough knowledge of the mathematical concept by teaching them in only one or two classes. It requires more idio-experience and idio-inspiration. So we could attain the above goal only by assimilating the senses of the mathematical concept and rethinking the nature of the concept into the subsequent processes of teaching, learning and solving problems.

Conclusion

We introduced, learned and discussed adequately a large number of foreign education theories after the reform. However, the practical study about the foreign education theories is weaker than other studies. A theory will be a rubber check if it can't be put in practice. So it is urgent to research the specific mathematics education practice in the sense of "action study".

A theory is good if it's helpful in solving practical problems. Recently, the "action study" is popular in the world wide and it is just centering on the specific education practice. It means that while not throw away theoretical study, we should cool down manufacturing or hunting new theories for novelty, and integrate the theories into the practices and practice according to the theories, without only saying " you have to do like this or like that" or labeling a certain teaching operation a creed. Zhang (2004) proposed that the practical study about mathematics education should be "up to mathematics, down to class" which is "modality conversion from academic math to educational mathematics." He gave a correct direction for the study on the mathematical education in our country.

References

Sawrey, J. M., & Telford, C. W. (1983). *Educational psychology*. Beijing: People's Education Press. ,8, 261-266.

Tu, R. (2003). Norm of research in math education. *Journal of Mathematics Education. 12,* 2-5.

Tu, R. (2003). Mathematics teaching epistemology. *Nanjing Normal University Research, 12,* 6-7.

Zhang, D. (2004). Education approaching "comprehensive mathematics" view. *Mathematics Education Study, 1,* 2-5.

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