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Building Place Value Understanding through Modeling and Structure

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Place value is a concept in which students in elementary school struggle and instruction and curricular materials continue to introduce and teach place value in a disconnected fashion. This study introduced place value through a modeling perspective, focusing specifically on using the bar model to represent units and quantity. The investigation piloted a place value module highlighting the use of the bar model in four first grade classrooms with high percentages of diverse learners, many from low-income families and with limited English language proficiency. The results indicated students successfully described the differences between units of 1 and 10 and could build and describe numbers in their teens and twenties. Students' vocabulary and understanding of place value improved over a three-week period, suggesting visual models can be used as an effective model to promote place value understanding.

Keywords: Place value, bar model, modeling, structure, early childhood.

Place value is a concept in which students in elementary school struggle, and instructional and curricular materials continue to introduce place value concepts in a fragmented manner. Understanding place value influences the understanding of other mathematical concepts such as number sense, rational number, and proportion (Fuson & Briars, 1990; McGuire & Kinzie, 2013). When addressing place value, one should be aware of the underlying features: relative position, unitizing, and language.

Most teachers are aware of *place value position* and most curricular materials highlight the ones, tens, hundreds, etc. In the U.S., students spend much time naming each digit's place. For instance, given the number 123, students will say the 2 is in the tens place. However, this does not necessarily mean they understand place value. Knowing the 2 represents two tens or twenty and that its relationship to units of 1 and 100 is critical (Chan, Au, & Tang, 2014).

This awareness of units is called *unitizing* and is a critical component to develop understanding and number sense (Fuson & Briars, 1990). When children fail to visualize this relationship or to see the relative sizes between units of place value, students operate with a procedural understanding of place value, not a conceptual understanding (Ross, 1989).

The English *language* might also be a barrier for young children beginning to conceptualize place value. Saying thirteen, for example, does not help the student imagine one ten and three ones. In many Asian countries researchers note students have stronger initial place value because their language for numbers between 10 and 20 is more explicitly connected to place value. For example, "ten-one" for eleven and "ten-two" for twelve (Miura, Okamoto, Kim, Steere, & Fayol, 1993).

This paper describes a study in which first graders were introduced to place value through a modeling perspective, focusing specifically on using the bar model to represent units and quantity (Ng & Lee, 2009; Van den Heuvel-Panhuizen, 2003). The study investigated the different ways students' understanding of place value changed by incorporating modeling and structure into pedagogical practices.

Conceptual Framework

One crucial role of teaching is to create equitable learning conditions that foster understanding so students can solve problems in many settings (Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Human, P., Murray, H., & Wearne, D., 1996). To do this, we used the Developing Mathematical Thinking (DMT) framework (Brendefur, 2008; Brendefur, Thiede, Strother, Bunning, & Peck, 2013), comprised of five critical dimensions: (a) taking student's ideas seriously, (b) pressing students conceptually, (c) encouraging multiple strategies and models, (d) addressing misconceptions, and (e) focusing on the structure of mathematics. These five dimensions frame an approach to teaching mathematics for understanding (Carpenter & Lehrer, 1999) incorporating notions of "progressive formalization" (Treffers, 1987). As Gravemeijer and van Galen (2003) describe, progressive formalization is a process of first allowing students to develop informal strategies and models to solve problems, and then, by critically examining both strategies and models, teachers press students to develop more sophisticated, formal, conventional and abstract strategies and procedures. By comparing solution strategies and examining the relationship among enactive, iconic and symbolic models (Brendefur, 2008; Bruner, 1964), students learn which manipulations make sense for given contexts and are encouraged to develop more generalizable procedures. In this paper, we highlight modeling, using mathematical representations and notations, as well as structure for developing a conceptual understanding place value.

Modeling is a key component of developing mathematical thinking. Knowledge originates from students' attempts to model or represent situations that can be mathematized and these initial models become the basis for solving related problems, as well as a means of support for more formal mathematical reasoning (Gravemeijer & van Galen, 2003). As Cobb (2000) describes, this use of modeling implies a shift in teaching where informal mathematical activity is used to support more formal mathematics. In classrooms, this is necessitated by the challenge to extend informal ideas to new situations. In this way, how we mathematize through representational models is a fundamental process in learning mathematics.

Progressive formalization is the process of formalizing students' ways of modeling through enactive, iconic, and symbolic representations without making huge leaps. This view of models and modeling contrasts with current practices in mathematics instruction in which models are used to "concretize expert knowledge" (Gravemeijer & van Galen, 2003, p. 118), such as when students are taught to model the traditional borrowing algorithm for subtraction with base-10 blocks. In contrast, the DMT module framework provides teachers with tasks that allow students to use, discuss, and practice moving among more informal and formal enactive, iconic, and symbolic models.

The focus of the DMT place value module is on building connections among mathematical strategies and models and to progressively formalize students' ideas and methods regarding place value. Discourse around mathematical ideas supports all students, including struggling learners (Brendefur & Frykholm, 2000; Moschkovich, 2012). By analyzing, comparing, and discussing different methods, students begin to conceptualize and then formalize their thinking. For example, first graders might initially solve a problem by using cubes, then represent the situation using a bar model or number line and then eventually use symbolic notations. By asking students to connect models, methods, and their thinking, a teacher can move students to effectively utilize new mathematical approaches involving mathematical formalization.

Focusing on structure allows students to build an understanding of and establish connections among fundamental concepts and particular topics being studied (National Council of Teachers of Mathematics, 2000). Structure, in this study, is defined as elements of the mathematics that remain constant across grade levels. For instance, the concepts of unit, equivalence and relationships and ways of interpreting these concepts, composing, decomposing, iteration, and partitioning are structural components for number. Understanding that the number 28 is composed of two units of size ten and eight units of size one is necessary to understand place value. For another example, by partitioning a single unit of one into ten equivalent size units, students learn to create a new unit of one-tenth, which iterates itself 10 time to compose one. Maintaining a focus on structure helps students internalize how foundational ideas extend across grade levels and topics. The module embeds the language of the structural components within each lesson's task and provides examples of how students might articulate and critique their own and others' mathematical models.

Methods

The study involved four Title 1 schools, where 77% of the students received free and reduced lunch. All four of the teachers were female with at least six years of teaching experience. One first grade teacher from each school was selected to teach the three-week module. Each of the teachers had attended a three-day workshop earlier in the summer on how to teach mathematics through the five DMT dimensions. This paper specifically highlighted one teacher's instruction and her students' modeling of place value over 10 lessons.

Similar data were collected from the four schools. We highlighted a representative set of students' work in one teacher's classroom. Students participated in all 10 module lessons. The module specifically incorporates modeling, structure and language (written and spoken) of the DMT framework.

Each lesson took the teachers one to two days to teach and focused on modeling through enactive, iconic, and symbolic representations to improve students' understanding of place value as evidenced through their drawings and language. Table 1 highlights the topics of the lessons. Elements of each lesson, and student work, are described in the discussion section.

Table 1
Module Lessons

Lesson	Focus
1	Unit of 1, iterating, bar model
2	Units of 1 and Units of 10, iterating, bar model, comparing units
3	Decomposing by units of place value
4	Tree diagrams and bar model comparisons
5	Correcting misconceptions and errors
6	Comparison problems
7	Problem solving
8	Modeling the context
9	Number lines, composing numbers
10	Number lines and language

Students kept a mathematics journal where the teachers cut and pasted problems for students to solve or asked them to construct bar models that represented different numbers. The journals were collected at the end of the three weeks and used as the primary data source. Two researchers visited and observed each teachers' classroom instruction twice throughout the study. A general inductive approach was used to evaluate the qualitative data (Thomas, 2006). In order to examine how teachers used student thinking, various sources of data were collected and analyzed using principles of qualitative research. Sources included interviews (audiotaped) and observations (audiotaped and field notes). In an iterative process, the data were coded and organized around an inductive approach to identify and understand themes and relationships within the data. General assertions were made through induction for each of these codes, which were then confirmed and disconfirmed by a third reader.

Results

Lesson 1: Initial Understanding of Place Value

Our goal was to provide first graders the opportunity to construct an initial understanding of place value through modeling and progressive formalization. In lesson 1, students learned to model a number by constructing bar models. Students used a single cube, which we named our "unit of 1" to iterate (copy) it 5 times with no gaps or overlaps to compose the number. The end result was an iconic representation, or bar model, of the number 5. To finish the model, we asked students to label the drawing. Figure 1 demonstrates Carol's modeling.

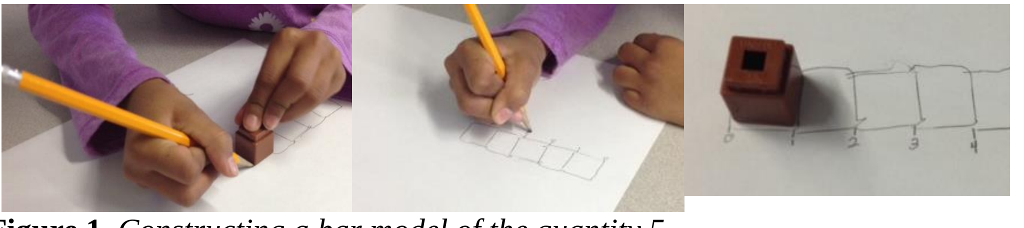

Figure 1. *Constructing a bar model of the quantity 5.*

Students were next asked to predict what their bar model would like for the quantities of 7, 12, 18, and 20. Most students said they would be longer, but the teacher pressed them to estimate how much longer by using their fingers. The idea was to begin thinking about 5 as a quantity to be iterated two or more times. Students constructed new bar models in a similar fashion to how the teacher demonstrated.

There were a few students, similar to the two children's work in Figure 2, whose iconic model needed more precision. Both students were able to use 1 cubes to construct the bar model. The teacher discussed with one of the students how to remove gaps and place each unit of 1 next another unit of 1. The other student was encouraged to use the cube and line up each mark.

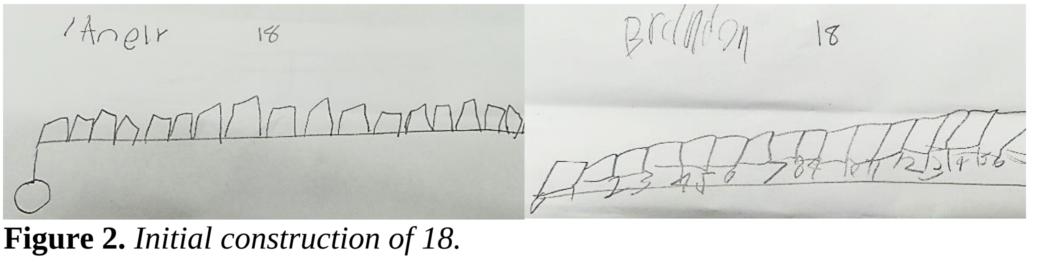

Figure 2. *Initial construction of 18.*

Lesson 2: Understanding Units of 1 and 10

After spending time building numbers by constructing bar models, tasks in lesson 2 focused on place value. During a whole class lesson, the teacher used units of 1 (cubes) to construct an enactive bar model of 10. Then, the teacher wrapped tape around the ten cubes and said, “we now have 1 unit of size 10.” She displayed the following Figure 3 to students.

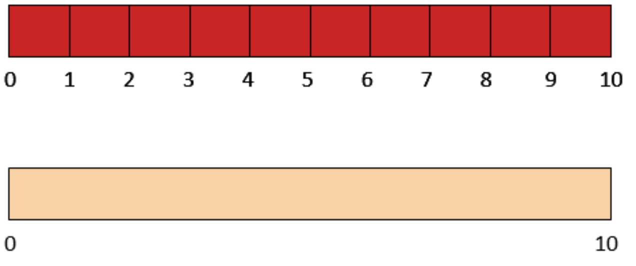

Figure 3. *Bar model representing 10 as ten units of 1 and 1 unit of 10.*

The teacher led the students through a discussion asking the following questions: What do you notice about the two different bar models? What is similar about them and what is different? The teacher shifted the conversation to focus the students on the concept that it takes 10 units of size 1 to get the same length of 1 unit of size 10. She asked students to practice telling partners: “This bar model is 10 units of size one. It is the same size as 1 unit of ten, but we are counting in different units.” “This bar model is 1 unit of ten. It is the same size as 10 units of one.” The teacher then constructed a word wall demonstrating a unit of 1, unit of 10, and the process of iterating.

After much practice, pointing to the iconic models and using this language, the students moved to constructing the number 12. Using enactive units of 1 and 10, each student constructed the number 12 iconically using 12 units of 1 and then with 1 unit of ten and 2 units of one. The teacher asked students to label their diagrams as is shown in Figure 4. Here the teacher highlighted and connected the symbolic notation to a) the iconic bar model, and b) the language:

> Teacher: Notice the first bar model has 12 units of one. Point to your model and say 12 units of 1. Now look at the second model. Point to the rod and say 1 unit of 10. How many units of one? [Student shout 2!] If we look at this number [10], the 1 represents the 1 unit of 10 [pointing] and the 0 represents zero units of 1. Here, [pointing to the 12], what does the digit 1 represent and what does the digit 2 represent?
> Avery: 1 unit of 10 and 2 ones.
> Teacher: Correct. Everyone turn to partners and explain what the digits in 10 and 12 mean.

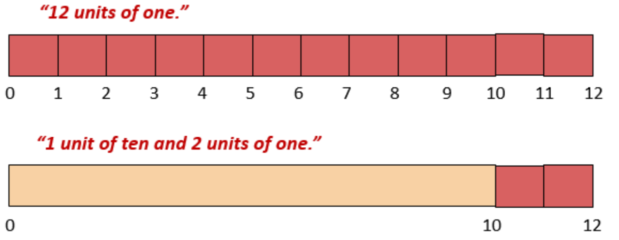

Figure 4. *Composing 12 with units of 1 and 10.*

The students continue this work by examining different numbers (15, 18, and 20).

Lesson 3: Enactive, Iconic and Symbolic Representations

Lesson 3 expanded on lesson 2 by focusing on units of 1 and units of 10 to model different number sets enactively (cubes and rods), iconically (bar model) and symbolically (numbers). Students were given sentences strips and challenged to use units of 1 and 10 to compose various two-digit numbers a variety of ways using only place value. For instance, students composed the number 36 with 36 units of 1, one unit of 10 and twenty-six units of 1, two units of 10 and sixteen units of one, or three units of 10 and six units of 1 (as demonstrated in Figure 5). Again, a word wall was used as a reference when using the structural language of units, iterate, compose and decompose. The written language in Figure 5 serves as an example.

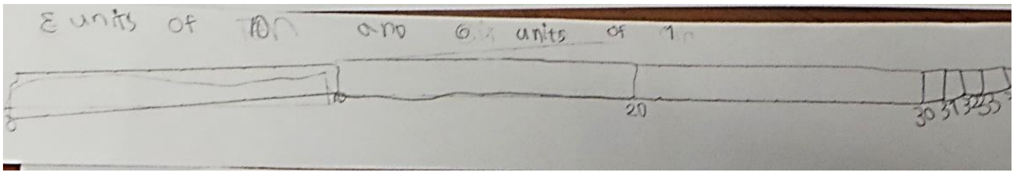

Figure 5. *Ariana's representation of 36*

Students' written response: Ɛ *(3)* units of ton *(ten)* and 6 units of 1

Lesson 4: Modeling Place Value with the Tree Diagram

Lesson 4 tasks were similar, but added the tree diagram as a symbolic model to represent numbers. Another student (see Figure 6) iconically displayed two units of 10 and three units of three. He labeled it correctly and used the tree diagram to demonstrate symbolically the relationships between the model and formal notation. He also showed the 3 represented three units of 1. Students continued building models enactively and iconically and then labeling the models symbolically. This was completed for the numbers 28, 36, 43, 48, 53 and 57.

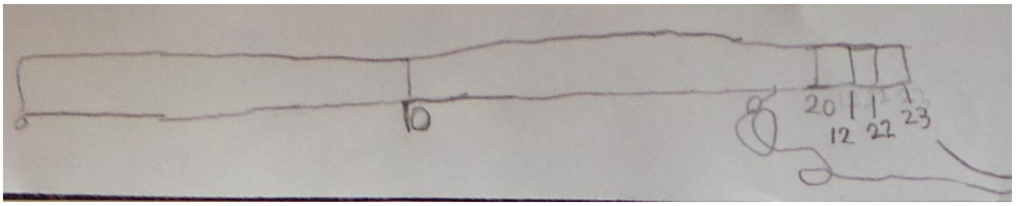

Figure 6. *Lian's modeling of 23.*

Lesson 5: Focusing on Misconceptions through Mistakes

In order to address potential student misconceptions, Lesson 5 focused on using bar models and tree diagrams to help students understand mistakes as listed in Table 2.

Table 2

Student's Incorrect Statements

Student	Number	Statement (mistake)
Edgar	24	"24 is 20 units of ten and 4 units of one."
Alice	41	"41 is 4 units of one and 1 unit of ten."
Steven	38	"38 is 2 units of ten and 28 units of one."

Students had to correctly model the numbers, and then stated specifically where Edgar, Alice and Steven made their mistakes.

Lesson 6: Comparison between Units of 1 and Units of 10

Lesson 6 focused on comparisons between two different numbers composed of units of 1 and 10. Figure 7 highlights two pairs of students comparing the numbers 43 and 31. Two other students, a boy and a girl, were able to iterate the unit of 10 and unit of 1 to create the numbers but mixed up the ones place and modeled 41 and 35. Their proportions were quite accurate and the teacher was able to work with them on explaining what the iconic bar model represented versus what they wrote down symbolically. The teacher encouraged them to attempt the next problem 18 and 35 (displayed in the second row). Here, the two were able to correctly model the quantities and find differences between them as noted with a yellow line that marked the 2, 10 and 5 and labeled 17.

In the third row, two additional students, another boy and another girl were able to model the numbers correctly and discuss the differences between the numbers: 1 unit of ten and 4 units of one, or 14. Students challenged themselves by representing and comparing numbers sets using the bar model and then used structural language to note the relative differences. Number sets began with 61 and 43 and extended to 114 and 44.

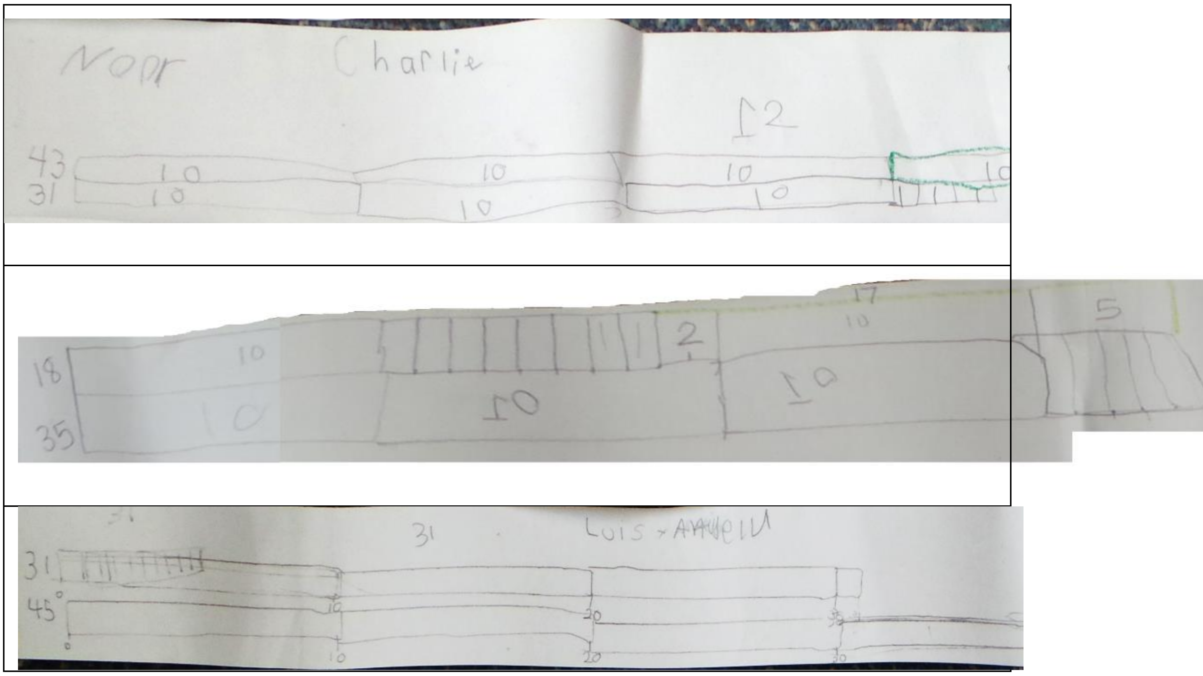

Figure 7. *Comparison of numbers 45 and 31.*

Lessons 7 & 8: Composing Numbers with Units of Ten and One within Contexts

Lessons 7 focused on using their skills and understanding of composing numbers with units of ten and one to solve problems in context. Students were asked to model the situation using iconic models and to describe how their model matched the situation. Task: Each shelf holds 10 books. How many shelves are full and how many books are left over if there are 34 books? Extension numbers were 65, 82, 103, and 125 books. In Figure 8, One female student used the bar model to represent 3 sets of 10 for the 30 books with 4 books left over. Lesson 8 continued with two more contextual problems each with 5 sets of numbers that increase in difficulty.

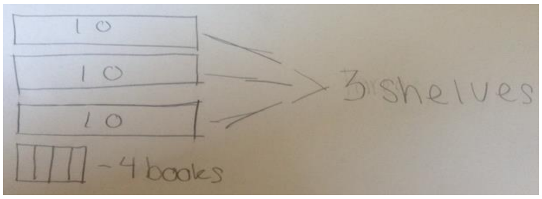

Figure 8. *Iconic representation of the book-shelf problem.*

Lesson 9: Progressively Formalizing Students' Work

The focus of lesson 9 was to progressively formalize students' work by introducing the number line as a way to model or represent the place value situation and then solve the expression. Students worked as a class to create a bar model of 34 using units of ten and one and then made jumps to compose this number. The transition is shown in Figure 9 where students, without support of the tools, composed the numbers by making jumps on the number line representative of proportional relationships between units. Three students' examples show how they did indeed progress to create proportional representations of the different numbers by focusing on units of ten and one.

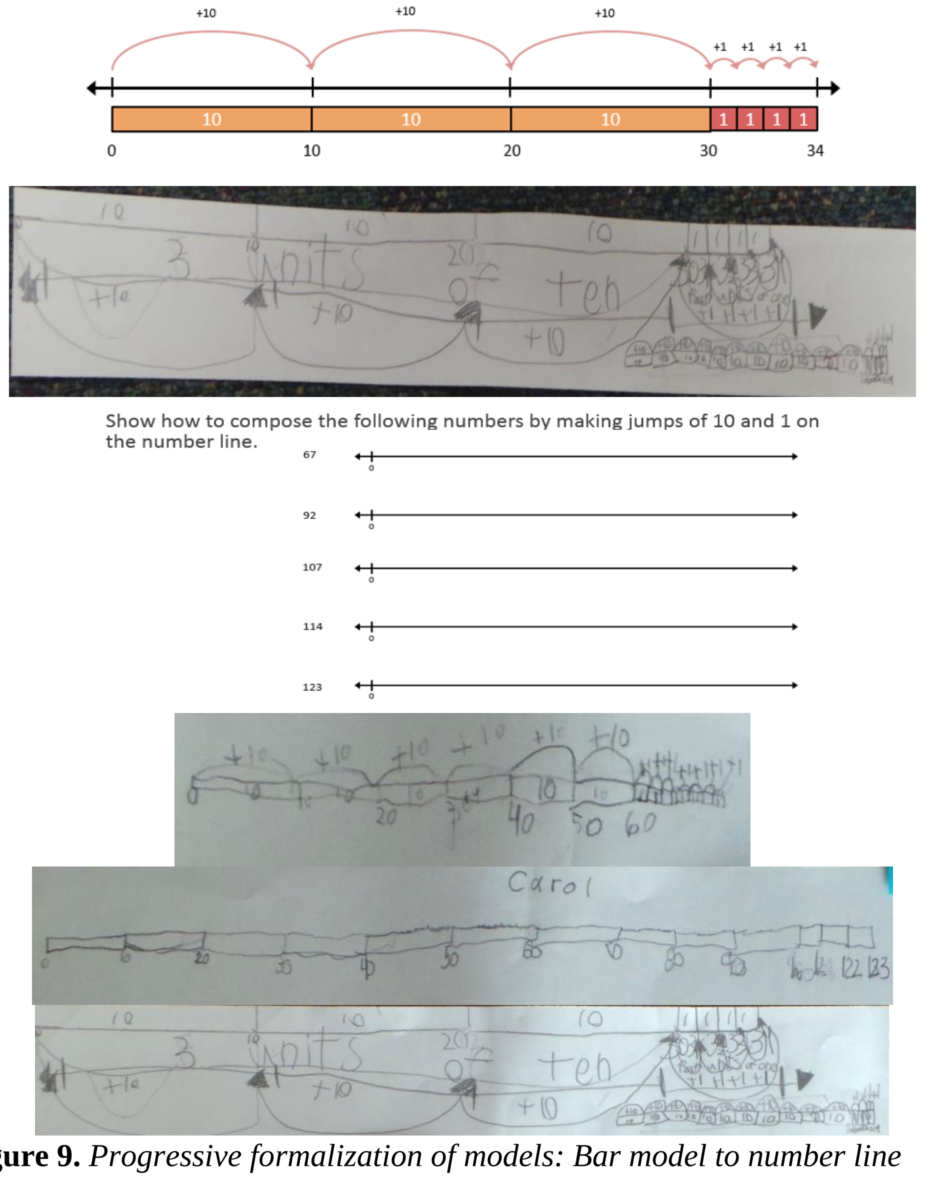

Figure 9. *Progressive formalization of models: Bar model to number line*

Lesson 10: Misconceptions and Language

Lesson 10 was the final lesson, which again pressed students to make sense of place value by focusing on other students' misconceptions. Students were asked to evaluate other students' comments about place value (see Table 3 below).

Table 3

Student Statements about Place Value

Student	Statements
Student 1	"I know that 74 is 70 tens and 4 ones."
Student 2	"I know that the only way to compose 125 is with 12 units of ten and 5 units of one."
Student 3	"98 and 89 are the same amount but you just write them differently."
Student 4	"13 units of ten and 7 units of one is more than 12 units of ten and 17 units of one."
Student 5	"200 has no units of ten or units of one. It is only 2 units of one hundred."
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Discussion and Conclusion

Five findings from this study highlight students' progression of understanding place value. More specifically, this study suggests that students were able to: (a) demonstrate quantities using visual representations and discuss differences in the relative size of the quantities, (b) construct and describe the differences between a number and the amounts each digit represented, (c) compare number lengths by their relative digit positions, (d) create proportional representations of numbers by place value; and, (e) use structure language to describe the processes of solving place value problems in context and describe the components of constructing visual representations.

The DMT approach suggests first graders were able to model and use structural language to discuss place value concepts (Brendefur et al, 2013). Unlike traditional approaches, students in this study were introduced to place value through linear situations and meaningful models to describe and notate units of 1 and 10. Initially, students used a measuring situation to construct numbers by iterating a cube, which represented a unit of 1, and constructing a bar model. This visual representation allowed students to begin visualizing quantity and use the language of composing and iterating. This is foundational to examining relationships and proportional size of numbers (Fuson & Briars, 1990; Ross, 1989).

Students were also able to describe the differences between units of 1 and 10 and could compose and describe numbers in their teens and twenties. Students focused on the relative position of the digits (e.g., 12 and 21) and constructed bar models that represented the appropriate number of units of 1 and 10. Again, their earlier work on the relative size or proportion of units of 1 and 10, allowed them to unitize multidigit numbers. This is critical for recognizing the difference in the placement of the digit 2 and 1 and their representational differences in quantity and proportion; this awareness is what Fuson and Briars (1990) calls unitizing. Ross (1989) described this as building a needed conceptual understanding in conjunction with procedural knowledge.

After working through the last few lessons, students' vocabulary included more structural language, which is a proxy for understanding place value. Here, we claim that linear models such as the bar model and number line allowed students to gain deeper knowledge about magnitude and proportion as related to place value. For example, focusing on iterating the spatial quantities of one and ten allowed 7 students to unitize in a similar way as to what Piaget found in the 1960s related to the conservation of length (Piaget, Inhelder, & Szeminska, 1960).

In addition, by having students create direct and indirect comparisons and by focusing on unit iteration and unit size in this study, we found first graders were able to describe with iconic models, symbols and language the key concepts of place value. We propose that using enactive models to build iconic models, while focusing on one unit of 1 and one unit of 10 to compose multi-digit numbers, and then connecting these iconic models to number notation and context, students builds a stronger foundation for understanding place value.

Moreover, the findings from our study addresses Baroody's (1990) question of how to effectively sequence the introduction of different models. Our results indicate that by incorporating modeling and structure into pedagogical practices students' understanding of place value increases in ways that build a foundation for learning future topics such as multi-digit place value, fraction and decimal understanding, and ratio and proportion. Future research is needed to determine whether learning place value as discussed in this article can be applied in different contexts, problem solving situations, and with upper grade students.

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