

Learning to See: Representing, Perceiving, and Judging Quantities of Numbers as a Task for In-Service Education for Pre-School Teachers

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In pre-school education there is now a focus on mathematics education, which especially applies for Germany. Because in Germany the new curricula came into operation only a few years ago, mathematics education has for many of the professionals not been a part of their own pre-service education. Therefore, it is quite important to set up in-service education with special focus on mathematics education. This paper illustrates one part of an in-service education for pre-school teachers with the focus on a special pedagogical mathematical content in the arithmetical domain: recognizing and supporting different processes which children can use to perceive and judge the quantity of numbers. German pre-schools are play-based, therefore, the professionals need to recognize children's competencies in everyday and play situations and then support children in developing mathematical competencies. This paper examines if and how the chosen pedagogical mathematical content is substantial for the professionals' diagnostic pedagogical content competencies.

Key words: *Pre-school teacher education, early mathematics education, pedagogical content knowledge, diagnostic pedagogical content knowledge, in-service education*

Many researchers have recognized that, by the time they enter school, children have already acquired powerful mathematical knowledge, skills, and dispositions (Baroody, 2010; Clarke, Clarke, & Cheeseman, 2006; Hunting & Mousley, 2010). This is true for most of the children. But it is also well acknowledged that there is a broad heterogeneity when children come to school (Schipper, 2009). Results of empirical research show, that there is a very strong correlation between the competencies of children at pre-school age and their success in mathematics later in school (Dornheim, 2008; Krajewski, 2008). Therefore, it is very important that professionals acquire professional knowledge or professional competencies so that they can recognize children's competencies and can support children in developing mathematical competencies already in pre-school institutions.

In-service teacher education for pre-school teachers is limited in time; it cannot substitute a whole pre-service education. Consequently, it has to be focused on selected tasks or themes that can provide a rich basis for learning opportunities for the pre-school teachers. In this paper it will be evaluated if the pedagogical content, representation, perception and judgment of quantities of numbers, constitutes a rich and possible task for pre-school teachers who never had any training in mathematics education.

Theoretical Background

There are many models that describe different aspects of professional knowledge or competencies of teachers in mathematics education (Shulman, 1986; Ball & Bass, 2003; Blömeke, Kaiser, & Lehmann, 2010; Lindmeier, 2011) as well as models of growth of professionals in mathematics education (Clarke & Hollingsworth, 2002). Three major categories of competencies can be found in nearly all of these models:

- (1) basic pedagogical content competencies,
- (2) diagnostic pedagogical content competencies,
- (3) mathematics instruction related action competencies.

These categories are also present in a paper of the Deutsche Mathematiker-Vereinigung (DMV), Gesellschaft für Didaktik der Mathematik (GDM) and Deutscher Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts (MNU) [Standards of teacher education for mathematics. Recommendation of DMV, GDM, MNU], a paper published in 2008 by these working groups (DMV, GDM, MNU, 2008).

Although these categories are not particularly designed with pre-school teachers in mind, they are useful for this paper. Indeed, various studies referring to pre-school mathematics highlight that these three aspects of professional competencies are important for pre-school mathematics too. McCray and Chen (2012) indicate that “a [pre-school] teacher must analyze and unpack the mathematical ideas embedded in the play scenario” (p. 295). Furthermore McCray and Chen (2012) refer to Clements, Sarama and DiBiase (2004) when they highlight the importance of content knowledge: “The ability to analyze a play situation and identify its opportunities for “mathematization” relies explicitly on teachers’ in-depth understanding of content knowledge” (p. 295). Gasteiger (2014) emphasizes also that pedagogical content knowledge is an important aspect of professional competencies for preschool teachers:

To implement early mathematics education in natural learning situations and to ensure that children with different levels of knowledge and skills can profit, early childhood educators need wide-ranging knowledge and competencies. First of all, they need content knowledge. They have to see the relations between

mathematics in the early years and later on to guarantee coherent mathematical learning (p. 278).

In addition to basic pedagogical content knowledge teachers also need diagnostic pedagogical content knowledge. Diagnostic competencies are necessary in particular for the implementation of early mathematics education in order to ensure that children with different levels of knowledge can benefit from early mathematics education. In German pre-schools (also known as kindergartens) early mathematics learning should be embedded in daily life and in play situations (Benz, Peter-Koop, & Größing, 2015); therefore pre-school teachers need the ability to

- see or identify opportunities for mathematization in daily life and in play
- see or identify levels of knowledge in children's actions and communications

These are the preconditions that pre-school teachers need to be able to interact with children in a supportive way and provide scaffolding through different kinds of instruction (Benz et al., 2015; Gasteiger, 2010, 2013; Lorenz, 2009, 2012). But in order to interact with children in this way, mathematics instruction related action competencies are necessary (Gasteiger, 2014).

These studies of pre-school teachers' competencies referring to early mathematics confirm that the mentioned competencies are also relevant for pre-school teachers. The main focus in this article is on the diagnostic pedagogical content competencies of pre-school teachers, because diagnostic pedagogical content competencies are the link between basic pedagogical content knowledge and mathematics instruction related action competencies. Basic pedagogical content knowledge is visible in diagnostic pedagogical content competencies, and diagnostic pedagogical content competencies constitute the basis for mathematics instruction related action competencies (Benz et al., 2015; Gasteiger, 2010; Lorenz, 2012)

In-service teacher education for pre-school teachers is limited in time; consequently, it has to be focused on selected tasks or themes, which, on the one hand can provide a rich basis for learning opportunities for pre-school teachers and guarantee coherent mathematical learning for the children on the other hand. The importance of different processes of representing, perceiving and judging numbers is also highlighted from a child's developmental perspective, so that children have a solid basis for coherent mathematical learning theories (Fritz, Ehlert & Balzer, 2013; Krajewski, 2008; Resnick, 1983, 1989) and empirical research of the content *perception and judgment of the quantity of numbers* (Benz, 2014; Clements, 1999; Gerster, 2009; Hess, 2012; Mulligan & Mitchelmore, 2009; Papic, Mulligan, & Mitchelmore, 2009, van Nes, 2009) show the importance of the different kind of processes

professionals can recognize in this area, so that they can support children to develop different competencies they need later on in school.

For most of the pre-school teachers, counting is a well accepted and important mathematical domain in pre-school mathematics education (Benz, 2010a). Research studies show very clearly that counting is a complex competence, which should be supported (Clements, 1984; Gelman & Gallistel, 1986; Fuson, 1988). But it is also obvious from theories of conceptual understanding of number that next to counting, part-whole reasoning is a very important aspect in building a broad conceptual understanding of numbers. Part-whole thinking and reasoning will be seen as an elaborated component of conceptual understanding of numbers (Baroody, 2010; Resnick, 1983).

The importance of the ability to compose and decompose quantities of numbers has been documented in studies with children at risk (Dornheim, 2008; Ellemor-Collins & Wright 2009, Gerster, 2009; Krakewski, 2008). Results show that flexible counting strategies, the ability to perceive structures in representation of quantities, and the perception of a quantity as a composition of two quantities are predictors for later success in mathematics (Dornheim, 2008). In order to structure quantities and to see parts of a whole, children must perceive structures in given representations. The ability to perceive structures of given representations of quantities and to use this perception for judging the number of the quantity can be seen as one part of part-whole thinking (Benz, 2014). Other studies suggest that the recognizing and perception of structures as one part of part-whole thinking is a good foundation for later success in mathematics (for empirical studies see Ellemor-Collins & Wright, 2009; Lüken, 2010; Mulligan & Mitchelmore, 2009; van Nes, 2009). These results suggest that using elaborated processes for identifying quantities of numbers, like perception of structures in quantities and using the structures for the judgment of the quantity of the numbers, is a competence which should be supported in pre-school already. Therefore, this pedagogical mathematical content was chosen as one main aspect for an in-service teacher training. So far it is not clear, if it is possible that the different aspects of the pedagogical mathematical content which is described above can be observed of pre-school teachers within an in-service training. Therefore, it will be examined *if* and *how* the selected pedagogical mathematical content can be a rich foundation for the development of diagnostic competencies especially for pre-school teachers through the following research question.

This paper examines which aspects of diagnostic pedagogical content competencies of pre-school teachers can be identified in the domain of processes used to perceive and judge quantities of representations. With this study the following research question was sought to answer:

What kind of processes of perceiving and judging a quantity of a representation of numbers can be observed by pre-school teachers?

Methodology

To investigate the research question, data of one part of an in-service teacher training course were analyzed. Although the course level is equivalent to an undergraduate level course, the pre-school teachers did not get an official academic grade for attending the in-service training. Here one part, *Counting and Seeing, is illustrated*, within the domain of numbers and sets. There were four other parts, *Exploring Patterns, Comparing and Measuring, Building and Placing, Finding Structures*. The in-service teacher training course lasted 3 years. Every part lasted six months and had three components:

- Component 1 - Workshop (5 hours). In the workshop it was focused on basic facts about the pedagogical content knowledge. In this workshop, learning and playing environments were created together with the pre-school teachers, in which children can acquire different abilities like counting, seeing structures and using structures to judge the quantity of a number. The results of the workshop were reported in a handout for the pre-school teachers, so that they could create these environments in their daily life in kindergarten.
- Component 2 – Pre-school teacher and pre-school children doing mathematics together (2hours). At the university a “laboratory or studio” was designed where a learning environment with different kinds of materials was provided (cf. Benz, 2010b). The pre-school teachers came to the university together with the group of children they normally work with. There, they had the possibility to explore mathematics together with their children. In this studio there were two student teachers in order to assist the pre-school teachers and in order to videotape the session. Only pre-school teachers who attended the workshop were allowed to come to the studio.
- *Meeting* (3 hours). The third component was a meeting with the pre-school teachers in order to reflect their experiences after they had explored mathematics together with their children in the studio. Video clips were the basis for this meeting. The scenes were selected on the basis of students’ and pre-school teachers’ reports about interesting situations during their visit at the studio. The meeting with pre-school teachers was audiotaped.

Participants could attend the course in each part only for the first two components or for all three components. In addition to participating of these three components the pre-school teachers were asked to help to collect data for a research project that was linked to the in-service program. They conducted a short one-to-one interview with their children with tasks about representation, perception, and judgment about numbers. They were also given descriptions of all materials, games, and the learning opportunities of these materials and games so that they can observe children and create learning opportunities for their children between the different components in their day-to-day work in kindergarten.

Subjects

In Germany, professionals working in kindergarten are called educators. They do normally not attend university for their pre-service education. Instead they usually attend 3 years in a school that provides vocational education, followed by a practical year. Because the in-service course was linked with a visit of the pre-school teachers together with the children of their pre-school, only teachers working in the area of Karlsruhe took part in the course.

The participation of the course was voluntary. Forty pre-school teachers took the first two components of the section *Counting and Seeing*. Nine of the forty pre-school teachers took part in the third component, the meeting. The reflections collected during the meeting portion of the program were audiotaped, and they are the basis for the data analysis. None of the pre-school teachers have had training in mathematics education prior to the in-service program; this is because in Germany the new curricula came into operation only a few years ago. Therefore mathematics education was for all of the professionals not a part of their own pre-service education. Five pre-school teachers had teaching experiences of 10 years or more and four pre-school teachers had teaching experiences between 5 and 10 years.

Method

During the meeting all conversations were recorded and transcribed in detail, and the transcripts served as basis for the analysis. The transcripts were divided in small semantic parts. Within this small semantic parts the passages of the transcripts were assigned to different categories with the deductive method of the qualitative content analysis (Mayring, 2007): The categories were deduced a priori from theoretical and other empirical research (Benz, 2011, 2014; Benz et al., 2015; Clarke et al., 2006; Clements et al., 2004; Dornheim, 2008; Fritz et al., 2013; Gerster, 2009; Peter-Koop & Grüßing, 2011; Peter-Koop, Wollring, Grüßing, & Spindeler, 2013). Two or three researchers analyzed all the passages of the transcripts and assigned the passages to different processes of representing, perceiving, and judging of quantities (Benz, 2014).

Results

Three selected situations will be reported in detail. The transcripts are illustrated as quotes or dialogs in order to show how the different passages of the text were referred to different categories of the diagnostic pedagogical content knowledge regarding different processes of representing, perceiving, and judging of quantities. After that, all categories which could be identified will be shown.

One pre-school teacher reported of a situation in which only for a short time two dice-patterns are shown (dice-pattern of 3 and 5) and the children should solve how many dots they have seen:

What I've seen from the children--those are insights I didn't have before. It already started with the dice pattern with 3 and 5 dots which we showed only for a very short time. It was very interesting how the children got the 8. Most of them said immediately 5 and 3. When I asked, how many dots are there together, it already begun. Some children calculated already, they just said the answer. And three children--really three children--painted patterns with the dots of the 5 and the 3 in the air and then counted them all in ones.

In this reflection the pre-school teacher reported that the children could recognize every single pattern of the dices in a way like *subitizing*. It cannot be decided if recognizing a pattern of a dice can be seen as the same cognitive act as subitizing of small quantities or if the children memorize the picture of the pattern as a whole figure. They could also just have learned the "name" of the figure without being aware of the quantity (von Glasersfeld, 1987). But in this case it is important that the pre-school teacher did realize that the children saw the different parts (dice pattern of 3 and 5) of the whole quantity immediately. Then the pre-school teacher realized different possibilities for judging the whole quantity: *knowing number facts* and *counting every single dot*. The interpretation that the pre-school teacher realized the process of *knowing number facts* is based on the statement of the quote "they just said the answer", the interpretation of *counting every single dot* is based on the statement "then counted them all in ones".

Another pre-school teacher reported about a situation in which the children played *buying and selling eggs*. In doing so the children used a rack containing 30 eggs in a carton that had 6 rows and 5 eggs per row (5x6). She commented her selected video clip as follows.

We played with the eggs, the egg cartons and egg racks. Ina put 30 eggs on the rack. And she started to count them all by 2s, 2, 4, 6, 8, 10, 12 and so on till 30. Then during playing--I don't know why--the children discussed how many eggs half of this egg rack may contain. Ina did a thing which I never would have realized if I had not been here. They still considered how many eggs will be in half of this egg carton. Then Ina draws an imaginary line in the carton so that there were 3 rows, each with 5 eggs. She laid her hand on 6 six eggs and said, "6" (Figure 1, left picture), then on other 6 eggs and said "12" (Figure 1, picture in the middle) and then on the last 3 eggs and said "15" (Figure 1, right picture). Because I didn't understand what was going on, I asked her to explain to me again what she did, and then we observed that her explanation will be on the video.

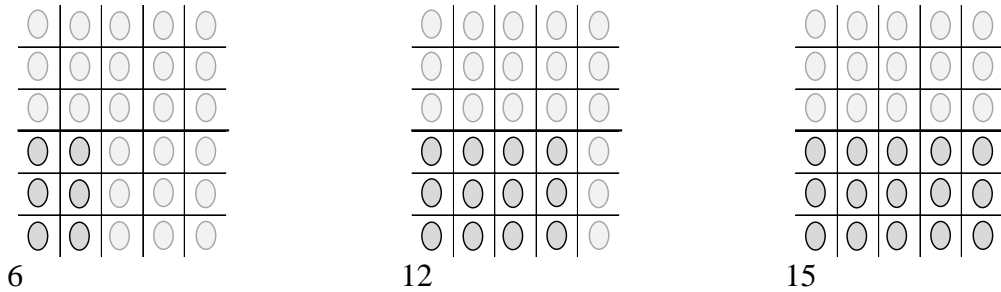


Figure 1. Ina's counting process.

Here again different perceptions of the quantity of numbers were reported, especially about *decomposing a quantity into different parts* and *using different structures*. First, the teacher reports that Ina decomposed the quantity of all eggs into parts of two eggs. Later, the pre-school teacher recognized that Ina decomposed all eggs into two halves. Then she observed that Ina decomposed the quantity of *half* of the eggs (15) in a structure with 2×6 eggs and 3 eggs. Next to these different possibilities of decomposition, the pre-school teacher also reported about different processes to judge the quantity like *counting in steps by twos* and about solving a very *challenging calculation task like 6 plus 6 plus 3*.

After this video-clip had been shown the pre-school teachers discussed that *seeing structures* is very individual and that the structures are not just there, they emerge when people are looking at them. The *individual act of perception of structures* was highlighted with another video-clip. One pre-school teacher told that she sat together with the children in a circle and that they discussed different arrangements of 5 eggs in an egg-carton, which has the structure of a ten-frame. The children wanted to make pairs with the same quantity in order to play memory. Most of the children put 5 eggs in a row, but Peter put 2 eggs in upper row and 3 eggs in lower row.

The following discussion can be seen in the videotape:

T: Why can you see easily and quickly that Peter has 5?

Child 1: Because 4 plus 1 is 5.

T: Hm, can you see how Peter put the eggs in the carton. Is that 4 and 1?

Child 1: Yeees?

T: Hmmmmm, yes, you could see that, too, right. But I've seen something else. Has anyone an idea?
(No child had an idea)

T: How many are in a row?

Child 3: 2 and 3.

T: Yes, and 2 and 3 is 5, too. Ok.

The teacher commented that only after the situation happened, she “understood how the child could see 4 and 1. Yes, it was the pattern of the dice of 4 and 1. But in this situation I just couldn't see it.”

Next to the problem that seeing structures is an individual act, the pre-school teachers discussed which questions were asked and how these questions were proposed. They discussed that the first question *Why can you see easily and quickly that Peter has 5* did help the children to focus on the structures of the representation. In this case it can be stated that pre-school teachers can recognize how the adults' questions can help to focus on the perception of decomposing a collection into different parts so that different kinds of perceiving and judging the quantity could be initiated. This is another example for mathematics instruction related action competencies. Although the pre-school teachers also discussed critically the way in which asking questions in this situation can be supportive and stimulate children's thinking this aspect is not analyzed in this paper.

Observed Categories for Representing and Judging Quantities

The analyses of the data generated five following categories of diagnostic pedagogical content knowledge associated with the pedagogical content *representation, perception, and judgment of quantities* of collections. See Figure 2.

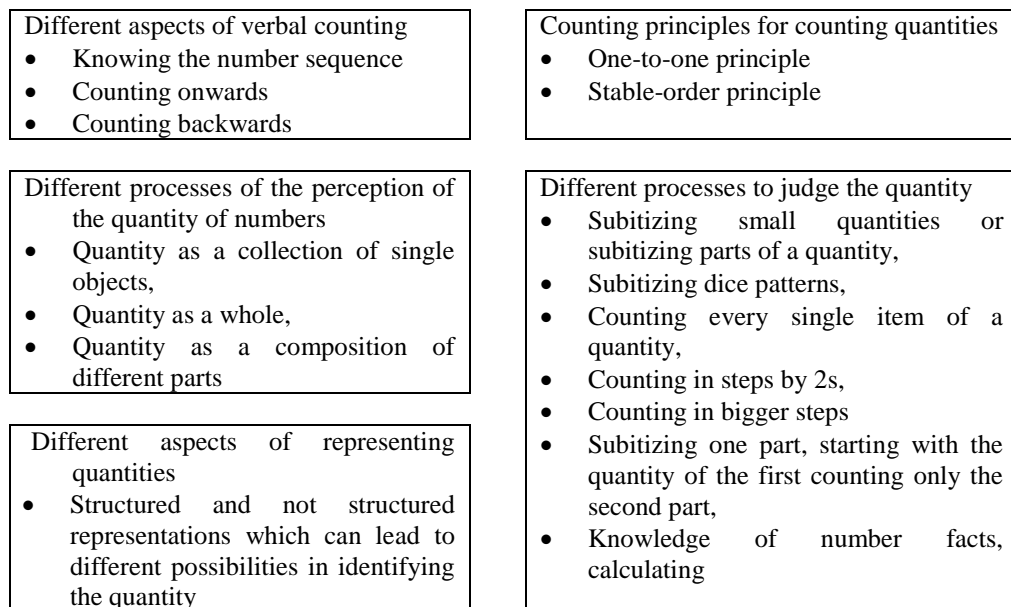


Figure 2. *Categories for representing, perceiving, and judging quantities.*

Comparing these different categories with the different possible processes in identifying the quantity of collections (Benz, 2011, 2014; Benz et al., 2015; Clarke et al., 2006; Clements et al., 2004; Fritz et al., 2013; Peter-Koop & Grüßing, 2011; Peter-Koop et al., 2013, Dornheim, 2008; Gerster, 2009), the pre-school teachers were able to observe different kinds of

processes and reported and described these processes in their observations. The pre-school teachers did not always use technical terms like *subitizing* or *part-whole-thinking* but they observed the processes and could describe them. For the category *subitizing* this can be shown in all three examples: In the first example the pre-school teacher reported that the children did know the numbers of the dice patterns *immediately* and in fact did subitize the patterns of the dice. In the second example the pre-school teacher reported that the child laid her hand on six eggs and said, “6”. The pre-school teacher did not observe a counting process and did not report about that. So here another observation of subitizing could be found. Also, in the third example the question why the children could see easily and quickly that Peter had 5 eggs in his egg carton showed that the pre-school teacher herself in the video focused on subitizing. Later on in the discussion about the video she commented about the child’s action that children did “see” the dice patterns of 4 and 1. This also can be interpreted that the pre-school teachers observed processes of subitizing. Similar to this interpretation of the data concerning the category *subitizing* for every category shown above, passages in the transcript of the audio taped meeting could be identified.

Discussion

It still has to be discussed to which extend the design of the in-service education contributes to the growth of the competencies of the pre-school teachers and influences the results. The pre-school teachers reported that the limited time of their attendance in the course and the length of the in-service education are important and influencing elements of the program. The pre-school teachers had time to develop and try different aspects in their daily routine in their kindergarten.

Still we must have in mind that the pre-school teachers who voluntarily attended in all three components of the in-service teacher training are very motivated and interested in mathematics education and are therefore perhaps not representative for all pre-school teachers.

The analysis reveals, that as a group, the pre-school teachers were able to observe and recognize *different kinds of representations* (e.g., structured and not structured representations, see, Dornheim, 2008; Gerster, 2009, Peter-Koop et al., 2013), which then can lead to different possibilities in identifying the quantity (Benz et al., 2015; Benz, 2014). The analysis also shows that the pre-school teachers have acquired content knowledge about different kinds of representations of quantities and that they can observe different kinds of representations in play situations (Examples 2 and 3). They could not only recognize different kinds of representations they also showed that they already had realized that seeing structures is an individual act (Example 3).

Concerning the content *perception of quantities* the pre-school teachers reported all different kinds of perception of quantities (Benz et al., 2015;

Benz, 2014). They reported that children sometimes perceived quantity as a collection of single objects when they counted them all in ones (this is not reported in the detailed examples but could be analyzed in the other passages). By reporting that children recognized dice patterns in Example 1, 2, and 3 the pre-school teachers observed children perceiving quantity as a whole. In Examples 2 and 3 pre-school teachers observed that children perceive quantities as compositions of different parts (different structures of the eggs in the egg racks and egg cartons).

The pre-school teachers recognized *different ways to judge the quantities* of numbers too (Benz, 2011, 2014; Benz et al., 2015; Clarke et al., 2006; Clements et al., 2004; Dornheim, 2008; Fritz et al., 2013; Gerster, 2009; Peter-Koop & Grüfing, 2011; Peter-Koop et al., 2013) and in doing so they also showed diagnostic competence regarding different aspects of verbal counting and some counting principles. In all three detailed examples the pre-school teachers reported about different processes of counting, different processes of subitizing, and of knowledge of number facts in order to calculate the quantity of several subitized parts.

Considering the design of the in-service education, and the special interest and motivation of this pre-school teachers, the analysis of the data, suggest that as a group the pre-school teachers can observe and recognize *all* the different and important aspects of the content *representation perception and judgment of quantities of numbers* identified by prior empirical research and existing theories in the domain. The analysis shows that pre-school teachers do observe the different aspects and categories of this pedagogical mathematical content.

Conclusion

This can now lead to two different conclusions.

The theoretical and normative aspects and categories concerning *representation perception and judgment of quantities of numbers* can be used as descriptive categories for diagnostic pedagogical content competencies of pre-school teachers. These aspects and categories can now be used in further empirical studies to evaluate the extent to which pre-school teachers have acquired diagnostic pedagogical content knowledge.

The chosen mathematical pedagogical content constitutes a rich task for pre-school teachers' education because in a limited time of training pre-school teachers can acquire the ability to recognize the different processes of the perception and judgment of the quantities of numbers. It also has to be considered that the observed processes are not only part of a conceptual understanding of numbers but that they constitute also different aspects of processes to solve addition and subtraction tasks.

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